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MAXIMALITY, LOCAL MAXIMALITY AND APPROXIMATION OF $\mathcal{F}$-PLURISUBHARMONIC FUNCTIONS

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SUMMARY OF MATHEMATICS DOCTOR THESIS

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- National Library of Viet Nam (Ha Noi)
- Library of Ha Noi National University of Education

Introduction

1. Rationale

The content of whole thesis studies a special class of plurisubharmonic functions. That is plurifine plurisubharmonic functions that we will write $\mathcal{F}$-plurisubharmonic. The history of problem gave study that started by H. Cartan 's results in the early 40s of last century. Then, to surmount discontinuity of subharmonic functions on $\mathbb{C}$, Cartan gave "fine" topology on $\mathbb{C}$ as weakest topology $\mathbb{C}$ that guarantees continuity of all subharmonic functions. He established some remarkable results with respect to above function class. Later on, in the 70s (of last century), Fuglede gave fine harmonic functions and fine holomorphic function and established relationship among them as relationship between fine harmonic function and fine holomorphic function in complex analysis book. Generalized above concepts to $\mathbb{C}^n$, Wiegerinck and partners built plurifine topology (one will denote by $\mathcal{F}$-topology) on $\mathbb{C}^n$ and defined the concept of $\mathcal{F}$-plurisubharmonic function. They developed the plurifine theory that one denotes $\mathcal{F}$-pluripotential.

A natural problem is arisen in $\mathcal{F}$-pluripotential theory is to investigate similar problem of normal pluripotential theory for the $\mathcal{F}$-plurisubharmonic function class.

As well-known, among the plurisubharmonic functions on a Euclidean open set Ω in $\mathbb{C}^n$, there exists a very important subclass, with many applications in pluripotential Theory, especially in the solution of the general Dirichlet problem, is the class of maximal plurisubharmonic functions. Thus, the study of the maximal of the plurisubharmonic function on the Euclidean open subset in $\mathbb{C}^n$ is one of the fundamental problems of the pluripotential Theory. Because the local maximal of the plurisubharmonic function are easier to see in many cases, a natural idea is change the consider of the maximal (global) of plurisubharmonic function to the consider local maximal of that function. However, until now, the radical solution of

the equivalence of local maximal of a arbitrarily plurisubharmonic function textit u on the open set Ω and the maximum of u on Ω is still an open problem.

Another issue that has been investigated recently by many authors is the approximation of the plurisubharmonic function by increasing sequence of plurisubharmonic functions on larger domain. Benelkourchi, Cegrell, Hed, Alevin, Persson, ... have been achieved excellent results about above problem in about ten years ago.

Following above approach, our thesis concentrate to study class of maximal $\mathcal{F}$-plurisubharmonic functions and the approximation problem of the $\mathcal{F}$-plurisubharmonic functions.

2. Objectives

Thesis concentrates to research some properties of $\mathcal{F}$-plurisubharmonic functions. More details, researching the relationship between local property and global property of the $\mathcal{F}$-plurisubharmonic functions, researching to establish approximation problem of the $\mathcal{F}$-plurisubharmonic functions by a increase sequence of normal plurisubharmonic functions.

3. Research subjects

- The plurisubharmonic function, $\mathcal{F}$-plurisubharmonic function and the maximal $\mathcal{F}$-plurisubharmonic function.
- The complex Monge-Ampère operator for the class of the finite $\mathcal{F}$-plurisubharmonic functions.
- Some function classes $\mathcal{F}$-plurisubharmonic on Ω: $\mathcal{E}_0(\Omega), \mathcal{F}_p(\Omega)$.
- Approximation problem of $\mathcal{F}$-plurisubharmonic functions.

4. Methodology

- Using some theory research methods in the basic mathematic research

with traditional tool and technique of the pluripotential theory, $\mathcal{F}$-pluripotential theory, functional analysis and the complex analysis.

- Participate in group seminars, organization department seminar to regularly exchange, discuss, disclose research results to receive information about scientific accuracy of research results in the community of specialized scientists.

5. Contributions of thesis

- The thesis pointed out the equivalence between the global $\mathcal{F}$-maximal property and the $\mathcal{F}$-locally $\mathcal{F}$-maximal property of the *continuous* $\mathcal{F}$-plurisubharmonic functions on $\mathcal{F}$-open sets in $\mathbb{C}^n$ (Theorem 2.1.2).
- Modifying above result and by new proof technique, Thesis pointed out the equivalence between global $\mathcal{F}$-maximal property and $\mathcal{F}$-locally $\mathcal{F}$-maximal property of the *bounded* $\mathcal{F}$-plurisubharmonic functions on $\mathcal{F}$-open sets in $\mathbb{C}^n$ (Theorem 2.2.2).

  This result has very scientific meaningful because it is true for the $\mathcal{F}$-plurisubharmonic functions on $\mathcal{F}$-open sets.
- Thesis gave concept of $\mathcal{F}$-hyperconvex domain and gave class $\mathcal{F}_p(\Omega)$. With such proper concepts, Thesis proved that $\mathcal{F}$-plurisubharmonic function can be approximated by an increase sequence of negative plurisubharmonic functions on decrease sequence of wider hyperconvex domains (Theorem 3.3.1).

6. The scientific and practical significance of the thesis

- Thesis given scientific result in thesis is new, updating with scientific meaning and moreover really contributed in studying property of $\mathcal{F}$-plurisubharmonic function.
- In terms of method, the thesis contributed to diversify system of research tool and technique in the complex analysis and pluripotential theory.

7. Research structure

Thesis' structure is presented according to concrete regulation with respect to the thesis of Ha Noi national University of Education. Structure of Thesis consists the parts: Introduction, Overview, Chapters, Conclusion, List of papers used in thesis, Reference.

The thesis' main content consists three chapters as follows.

- **Chapter 1. $\mathcal{F}$-Plurisubharmonic function and the complex Monge-Ampère operator**

  Chapter 1 presents some necessary elements about $\mathcal{F}$-topology, definition and some properties of $\mathcal{F}$-Plurisubharmonic function and the complex Monge-Ampère operator and maximal $\mathcal{F}$-Plurisubharmonic function, also some results that will be used in next chapters.

- **Chapter 2. Local property of maximal $\mathcal{F}$-plurisubharmonic function**

  Chapter 2 gave *continuous* or *bounded* condition to ensure necessary and sufficient condition to a $\mathcal{F}$-plurisubharmonic function, that $\mathcal{F}$-locally $\mathcal{F}$-maximal, is global $\mathcal{F}$-maximal on $\mathcal{F}$-open set Ω in $\mathbb{C}^n$. The main results obtained are Theorem 2.1.2 and Theorem 2.2.2.

- **Chapter 3. Approximation of $\mathcal{F}$-plurisubharmonic function**

  Chapter 3 pointed out when negative $\mathcal{F}$-plurisubharmonic function u in $\mathcal{F}$-domain Ω that can be approximated by increase sequence of negative plurisubharmonic functions defined on Euclidean neighborhoods of Ω. The main result of this chapter is Theorem 3.3.1.

  In the last part, in conclusion, we recall the main research results presented in Thesis. In request part, we boldly give some next research ideas to develop the thesis' topic. We hope that will receive many sharing attentions of scientists and colleagues to complete the research results.

Overview of research problems in thesis

As presented above, the plurisubharmonic function is one of central objects of the pluripotential theory. Investigating the plurisubharmonic functions, that its focus researches the complex Monge-Ampre operators, attracted attention of many big mathematicians from the decade 30s of the century XX and achieved profound results. These works of authors K. Oka, H. Cartan, P. Lelong, E. Bedford, B.A. Taylor, U. Cegrell, S. Kolodziej and so on not only profoundly influence to development of complex analysis in several variables in particular, but also promoted the development of many other fields in modern mathematics.

According to normal Euclidean topology, plurisubharmonic functions generally not continuous, while continuity keeps central role in investigating functional theory. Therefore, giving new topologies to describe better continuity of the plurisubharmonic function attracted attention from the decade 30s of the former century.

In 2003, El Kadiri defined the concept of $\mathcal{F}$-plurisubharmonic functions on $\mathcal{F}$-open subset of $\mathbb{C}^n$ and studied the properties of those functions. these functions introduced as upper $\mathcal{F}$-semicontinuous functions that restriction on the complex lines is $\mathcal{F}$-subharmonic function, where a $\mathcal{F}$-subharmonic function defined on a $\mathcal{F}$-domain is upper semicontinuous and satisfies mean value inequality. This definition is naturally extension plurisubharmonic function for $\mathcal{F}$-plurisubharmonic functions.

In 2010, El Marzguioui and Wiegerinck investigated continuous property of $\mathcal{F}$-plurisubharmonic functions on $\mathcal{F}$-open set. They proved that $\mathcal{F}$-plurisubharmonic function is $\mathcal{F}$-continuous.

In 2011, Kadiri, Fuglede and Wiegerinck proved many important properties of the $\mathcal{F}$-plurisubharmonic function.

In 2004 and 2006 on a bounded hyperconvex domain (resp. on an open set) $\Omega \subset \mathbb{C}^n$, Cegrell, resp. Bocki constructed the complex Monge-Ampère

operator with respect to a class of unbounded plurisubharmonic functions. Moreover, in the same form with respect to a class of locally bounded plurisubharmonic functions, they proved that maximum is a local concept with respect to the classes of plurisubharmonic functions that they introduced and investigated.

In 2014, El Kadiri and Wiegerinck defined Monge Ampre operator on finite $\mathcal{F}$-plurisubharmonic functions in $\mathcal{F}$-open sets and pointed out that it was defined as the positive measure. El Kadiri and M. Smit introduced and researched the concept of $\mathcal{F}$-maxinmal $\mathcal{F}$-plurisubharmonic function and the $\mathcal{F}$-locally $\mathcal{F}$-maxinmal $\mathcal{F}$-plurisubharmonic functions, that is extension of the concept of maximal plurisubharmonic function on an Euclidean domain to a $\mathcal{F}$-domain of $\mathbb{C}^n$ in the natural way. They proved that each bounded $\mathcal{F}$-locally $\mathcal{F}$-maxinmal plurisubharmonic function, that define on an Euclidean open set, is $\mathcal{F}$-maxinmal and they gave an example to show that this result is not valid when the function is not finite.

The first research direction of Thesis is to extend the result of above authors with respect to $\mathcal{F}$-plurisubharmonic functions. Concretely, we study sufficient condition to receive $\mathcal{F}$-maximal property of $\mathcal{F}$-plurisubharmonic functions on $\mathcal{F}$-open sets from local property respectively.

Next we consider the problem approximation of $\mathcal{F}$-plurisubharmonic functions by increasing sequence of plurisubharmonic functions. The history of this problem is as follows:

The first result belongs to Fornss and Wiegerinck, gave theorem (1989), confirmed that, if Ω is a bounded domain with C^1-boundary and u continuous on $\overline{\Omega}$ then u can be approximated uniformly on $\overline{\Omega}$ by a sequence of smooth plurisubharmonic functions defined on Euclidean neighborhoods of Ω. Recently, Avelin, Hed, and Persson extended this result to domains with local boundaries given by graphs of continuous functions. Moreover, According to the results given by S. Benelkourchi, Cegrell, L.Hed and N.X. Hong, the plurisubharmonic functions u can be approximated

monotonously from outside by increasing sequence of plurisubharmonic functions, if the domain Ω has the $\mathcal{F}$-approximation property and u belongs to one of the Cegrells classes in Ω.

Above mentioned results leads to following problem: let u be a negative $\mathcal{F}$-plurisubharmonic function in $\mathcal{F}$-domain Ω. when u can be approximated by a increase sequence of plurisubharmonic functions defined on Euclidean neighborhoods of Ω?

Therefore in this thesis, we investigates and solves two following problems.

The first problem: Study The local property of the maximal $\mathcal{F}$-plurisubharmonic function

Klimek proved that a locally bounded $\mathcal{PSH}$ function u defined on an Euclidean open set is maximal if and only if $(dd^c u)^n = 0$, and therefore, the bounded $\mathcal{PSH}$ function defined on an Euclidean open set is maximal if and only if it is locally maximal. Therefore, maximal property and locally maximal property with respect to bounded plurisubharmonic function defined in defined on an Euclidean open set is equivalent.

Although with respect to a bounded $\mathcal{F}$-$\mathcal{PSH}$ function u defined on an $\mathcal{F}$-open Ω, the complex MongeAmpre operator $(dd^c u)^n$ can be $\mathcal{F}$-locally defined on Ω, however normal techniques of Klimek and authors can't apply for situation that u is $\mathcal{F}$-plurisubharmonic function on Ω. Hence, we need to find another techniques to generalize Klimek's result for the class of $\mathcal{F}$-plurisubharmonic functions.

As mentioned above, we give sufficient conditions so that from $\mathcal{F}$-locally $\mathcal{F}$-maximal property of a $\mathcal{F}$-plurisubharmonic function on a $\mathcal{F}$-open set Ω in $\mathbb{C}^n$ also is $\mathcal{F}$-maximal on Ω. Concretely: Theorem 2.1.2 pointed out that with respect to a continuous $\mathcal{F}$-plurisubharmonic function on Ω, this function is $\mathcal{F}$-maximal on Ω if and only if it is $\mathcal{F}$-locally $\mathcal{F}$-maximal on Ω.

Next, we extend the above result by replacing the condition *continuous* in the Theorem 2.1.2 by the weaker condition that is *bounded* of

$\mathcal{F}$-plurisubharmonic function in a $\mathcal{F}$-open set Ω in $\mathbb{C}^n$. Result achieved is Theorem 2.2.2 pointed out that $\mathcal{F}$-maximal property is equivalent with $\mathcal{F}$-locally $\mathcal{F}$-maximal property.

The second problem: Study approximation of $\mathcal{F}$-plurisubharmonic function by the plurisubharmonic functions

We give sufficient conditions to a $\mathcal{F}$-plurisubharmonic function u in an Euclidean open subset Ω approximated by a increase sequence of plurisubharmonic functions. Where, approximation is understood in the sense of u can be approximated uniformly in $\overline{\Omega}$ by a increase sequence of smooth plurisubharmonic functions defined on Euclidean neighborhoods of Ω.

By giving concept of $\mathcal{F}$-hyperconvex domain and defining the class $\mathcal{F}$-$\mathcal{PSH}$ functions $\mathcal{E}_0(\Omega)$ and $\mathcal{F}_p(\Omega)$, we proved Theorem 3.3.1, where state that each function $u \in \mathcal{F}_p(\Omega)\,(p > 0)$ always approximate by an increase sequence of plurisubharmonic functions in neighborhood of Ω.

The results of the thesis have been reported at the conference, workshop:

[1] Conference (01/2017), "*Approximation of $\mathcal{F}$-plurisubharmonic functions*", Conference of Scientific Research Report, Faculty of Mathematics - Informatics, Hanoi University of Education.

[2] Conference (12/2017), "*Local maximality for bounded $\mathcal{F}$-plurisubharmonic functions*", Conference of Scientific Research Report, Faculty of Mathematics - Informatics, Hanoi University of Education.

[3] Conference of Scientific (8/2018), "*Local property of maximal $\mathcal{F}$-$\mathcal{PSH}$ functions*", Mathematical Analysis Report - Vietnam Mathematical Congress IX - Nha Trang.

Chapter 1

$\mathcal{F}$-Plurisubharmonic function, Maximal $\mathcal{F}$-plurisubharmonic functions, and the complex Monge-Ampère Operator

In this chapter, we recall some basic elements about the $\mathcal{F}$-topology in $\mathbb{C}^n$, the $\mathcal{F}$-plurisubharmonic function, the complex Monge-Ampère operator for $\mathcal{F}$-Plurisubharmonic functions, the maximal $\mathcal{F}$-plurisubharmonic function and give some research results that used in Thesis.

1.1 $\mathcal{F}$-topology and $\mathcal{F}$-plurisubharmonic function

Next, we recall some basic knowledge about $\mathcal{F}$-topology given by E. Bedford, B. A. Taylor, El. Marzguioui and J. Wiegerinck.

Definition 1.1.1. The *$\mathcal{F}$-topology* on a Euclidean open set $\Omega \subset \mathbb{C}^n$ is the smallest topology on Ω that makes all plurisubharmonic functions on Ω are continuous.

Since plurisubharmonic functions are always upper semicontinuous, a

local subbase at any $a \in \Omega$ given by sets

$$U(a, \mathbb{B}, \varphi) = \{z \in \mathbb{B} : \varphi(z) > 0\}$$

where $\mathbb{B} \subset \Omega$ is a ball with center at a and $\varphi \in \mathcal{PSH}(\mathbb{B})$ that $\varphi(a) > 0$.

The following Proposition describes neighborhood of the point a.

Proposition 1.1.2. *Let $\mathcal{F}$-topology $\mathcal{F}$ on a Euclidean open set $\Omega \subset \mathbb{C}^n$. Then, the sets $U(a, \mathbb{B}, \varphi)$ form a local basis for $\mathcal{F}$-topology $\mathcal{F}$.*

The following Theorem was given by J. Wiegerinck about property of $\mathcal{F}$-topology $\mathcal{F}$.

Theorem 1.1.3. *Let $\mathcal{F}$-topology $\mathcal{F}$ on a Euclidean open set $\Omega \subset \mathbb{C}^n$.*

i) $\mathcal{F}$ is quasi-Lindelöf, that is, every arbitrary union of $\mathcal{F}$-open sets is the union of a countable subunion and a pluripolar set.

ii) $\mathcal{F}$ is completely regular, that is, for every $\mathcal{F}$-closed set $A \subset \Omega$ and $a \in \Omega \backslash A$, there exists an $\mathcal{F}$-continuous function f such that $f|_A = 0$ and $f(a) \neq 0$.

Remark 1.1.4. Euclidean topology smaller $\mathcal{F}$-topology.

Next, we recall some definitions, propositions given by J. Wiegerinck, M. El Kadiri and M. Smit.

Definition 1.1.5. A function f defined on a $\mathcal{F}$-open set $U \subset \mathbb{R}^n$ is called *$\mathcal{F}$-subharmonic* if:

(i) f is $\mathcal{F}$-upper semicontinuous;

(ii) $f(z) \leqslant \int_{\partial_{\mathcal{F}} V} f d\delta_z^{U \backslash V}$ for V in some local base of the $\mathcal{F}$-topology at z;

(iii) $f \not\equiv -\infty$ on every $\mathcal{F}$-component of U.

Definition 1.1.6. Let Ω denote an $\mathcal{F}$-open subset of $\mathbb{C}^n$.

A function $f : \Omega \longrightarrow [-\infty, +\infty)$ is said to be $\mathcal{F}$-plurisubharmonic if f is $\mathcal{F}$-upper semicontinuous and, for every complex line l in $\mathbb{C}^n$, the restriction of f to any $\mathcal{F}$-component of the $\mathcal{F}$-open subset $l \cap \Omega$ of l is either $\mathcal{F}$-subharmonic or $\equiv -\infty$.

Proposition 1.1.7. *Let G and U is $\mathcal{F}$-open sets on $\mathbb{C}^n$ that $G \subseteq U$. Assume $u \in \mathcal{F}\text{-}\mathcal{PSH}(U)$, $v \in \mathcal{F}\text{-}\mathcal{PSH}(G)$, and $\mathcal{F}\text{-}\limsup_{G \ni z \to \zeta} v(z) \leqslant u(\zeta)$ for all $\zeta \in \partial_{\mathcal{F}} G \cap U$. Then, function*

$$\omega(z) = \begin{cases} \max(u(z), v(z)) & \text{if } z \in G, \\ u(z) & \text{if } z \in U \backslash G, \end{cases}$$

in $\mathcal{F}\text{-}\mathcal{PSH}(U)$.

Next, one recalls some properties of $\mathcal{F}$-plurisubharmonic functions by M. El Kadiri, B. Fuglede, J. Wiegerinck.

Proposition 1.1.8. *Let Ω be a Euclidean open subset of $\mathbb{C}^n$.*

For a function $f : \Omega \to [-\infty; +\infty)$, the following are equivalent:

(i) f is plurisubharmonic (in the ordinary sense).

(ii) f is C-strongly $\mathcal{F}$plurisubharmonic and not identically $-\infty$ on any component of Ω.

(iii) f is $\mathcal{F}$plurisubharmonic and not identically $-\infty$ on any component of Ω.

We speech and prove two following propositions that used in Chapter 3.

Proposition 1.1.9. *Let Ω be an $\mathcal{F}$-open set in $\mathbb{C}^n$ and let $u \in \mathcal{F}\text{-}\mathcal{PSH}^-(\Omega)$. Assume that $\chi : \mathbb{R}^- \to \mathbb{R}^-$ is an increasing convex function. Then*

$$\chi \circ u \in \mathcal{F}\text{-}\mathcal{PSH}^-(\Omega).$$

Proposition 1.1.10. *Let Ω be an $\mathcal{F}$-open set in $\mathbb{C}^n$ and let φ be a strictly plurisubharmonic function on $\mathbb{C}^n$. Assume that $u, v \in \mathcal{F}\text{-}\mathcal{PSH}^-(\Omega)$ such that*

$$\int\limits_{\Omega \cap \{-\infty < u < v\}} (dd^c \varphi)^n = 0.$$

Then $u \geq v$ on Ω.

1.2 Complex Monge-Ampère operator

Firstly, we recall two definitions and two following theorems given by M. El Kadiri and J. Wiegerinck.

Definition 1.2.1. We denote $QB(\mathbb{C}^n)$ by σ-algebra on $\mathbb{C}^n$ generated by Borel sets and pluripolar subsets of $\mathbb{C}^n$, $QB(U)$ by restriction of $QB(\mathbb{C}^n)$ on U.

Theorem 1.2.2. *Let u be an $\mathcal{F}$-plurisubharmonic function on U. Then there exists a pluripolar $\mathcal{F}$-closed set E trong U such that the u is C-strongly $\mathcal{F}$-plurisubharmonic on $U\backslash E$.*

Theorem 1.2.3. *Suppose that u_1, u_2, v_1, v_2 are plurisubharmonic function on a domain $\Omega \subset \mathbb{C}^n$. If $u_1 - u_2 = v_1 - v_2$ on an $\mathcal{F}$-open $\mathcal{O} \subset \Omega$, then $(dd^c(u_1 - u_2))^n|_{\mathcal{O}} = (dd^c(v_1 - v_2))^n|_{\mathcal{O}}$.*

From the result above, we can defined that the Monge-Ampère operator for $\mathcal{F}$-plurisubharmonic functions be finite. We have

Definition 1.2.4. Let Ω be an $\mathcal{F}$-open set in $\mathbb{C}^n$ and let $QB(\Omega)$ be the trace of $QB(\mathbb{C}^n)$ on Ω. Assume that $u_1, \ldots, u_n \in \mathcal{F}\text{-}\mathcal{PSH}(\Omega)$ be finite. Using the quasi-Lindelöf property of the plurifine topology and Theorem 1.2.2, there exist a pluripolar set $E \subset \Omega$, a sequence of $\mathcal{F}$-open subsets $\{O_k\}$ and plurisubharmonic functions $f_{j,k}, g_{j,k}$ defined on Euclidean neighborhoods of $\overline{O}_k$ such that $\Omega = E \cup \bigcup_{k=1}^{\infty} O_k$ and $u_j = f_{j,k} - g_{j,k}$ on O_k. We define $O_0 := \emptyset$ and

$$\begin{aligned} &\int\limits_A dd^c u_1 \wedge \ldots \wedge dd^c u_n := \\ &\sum_{j=1}^{\infty} \int\limits_{A \cap (O_j \backslash \bigcup_{k=0}^{j-1} O_k)} dd^c(f_{1,j} - g_{1,j}) \wedge \ldots \wedge dd^c(f_{n,j} - g_{n,j}), \end{aligned} \tag{1.1}$$

$A \in QB(\Omega)$. By Theorem 1.2.3, formula (1.1) is defined a the measure on E.

Note that, this measure does independent on $\{O_k\}$, $\{f_{j,k}\}$ and $\{g_{j,k}\}$. I say it is *the complex Monge-Ampère measure* of $\mathcal{F}$-plurisubharmonic functions .

Following, one recalls proposition given by E. Bedford and B. A. Taylor.

Proposition 1.2.5. *Let Ω be an $\mathcal{F}$-open set in $\mathbb{C}^n$ and let $u_1, \ldots, u_n \in \mathcal{F}$-$\mathcal{PSH}(\Omega)$ be finite. Then, $dd^c u_1 \wedge \ldots \wedge dd^c u_n$ is a non-negative measure in $QB(\Omega)$.*

Next, we give some results through two propositions that be used in Chapter 3.

Proposition 1.2.6. *Let Ω be an $\mathcal{F}$-open set in $\mathbb{C}^n$ and let μ be a non-negative measure on $QB(\Omega)$. Assume that $u, v \in \mathcal{F}$-$\mathcal{PSH}(\Omega)$ are finite such that $(dd^c u)^n \geq \mu$ and $(dd^c v)^n \geq \mu$ in Ω. Then*

$$(dd^c \max(u, v))^n \geq \mu \text{ in } \Omega.$$

Following result presents quasi-continuity of Monge-Ampère operator and the monotone sequences of $\mathcal{F}$-plurisubharmonic functions.

Proposition 1.2.7. *Let Ω be an $\mathcal{F}$-open set in $\mathbb{C}^n$ and let $u \in \mathcal{F}$-$\mathcal{PSH}^-(\Omega)$ be finite. Assume that $\{u_j\}$ is a monotone sequence of negative, finite, $\mathcal{F}$-plurisubharmonic functions such that $u_j \to u$ a.e. on Ω. Then*

$$\int_\Omega f(dd^c u)^n \leq \liminf_{j \to +\infty} \int_\Omega f(dd^c u_j)^n,$$

for every non-negative, bounded, $\mathcal{F}$-continuous function f on Ω.

1.3 Maximal $\mathcal{F}$-plurisubharmonic functions

Now, we introduce the $\mathcal{F}$-plurisubharmonic functions class have many utilities in $\mathcal{F}$-pluripotential Theory.

Following definition was given by M. El Kadiri and M. Smit.

Definition 1.3.1. Let $\Omega \subset \mathbb{C}^n$ be an $\mathcal{F}$-open and let $u \in \mathcal{F}\text{-}\mathcal{PSH}(\Omega)$. We say that u is *$\mathcal{F}$-maximal* on Ω if for every bounded $\mathcal{F}$-open G of $\mathbb{C}^n$ for $\overline{G} \subset \Omega$, and for every function $v \in \mathcal{F}\text{-}\mathcal{PSH}(G)$ that is bounded from above on G and extends $\mathcal{F}$-upper semicontinuously to $\overline{G}^{\mathcal{F}}$ with $v \leqslant u$ on $\partial_{\mathcal{F}} G$ implies $v \leqslant u$ on G.

In 2014, El Kadiri and Smit proved some basic properties of maximal $\mathcal{F}$-plurisubharmonic function. By following proposition, they pointed out that a necessary condition to a $\mathcal{F}$-plurisubharmonic function is $\mathcal{F}$-maximal.

Proposition 1.3.2. *Let f be a finite $\mathcal{F}$-maximal $\mathcal{F}$-$\mathcal{PSH}$ on an $\mathcal{F}$-domain U in $\mathbb{C}^n$. Then we have $(dd^c f)^n = 0$.*

Important results about maximal $\mathcal{F}$-plurisubharmonic function are stated by us through two following propositions.

Proposition 1.3.3. *Let Ω be an $\mathcal{F}$-open set in $\mathbb{C}^n$. Assume that u is a bounded $\mathcal{F}$-plurisubharmonic function on Ω.*

Then, the following conditions are equivalent:

(a) $u \in \mathcal{F}\text{-}\mathcal{MPSH}(\Omega)$.

(b) $u + g \in \mathcal{F}\text{-}\mathcal{MPSH}(\Omega)$, for every pluriharmonic functions g on $\mathbb{C}^n$.

(c) For every $v \in \mathcal{F}\text{-}\mathcal{PSH}(\Omega)$ and for every $\mathcal{F}$-open set $G \subset \Omega$ with $\overline{G} \subset \Omega$, we have

$$\sup_G (v - u) \leq \sup_{\Omega \setminus G} (v - u).$$

Proposition 1.3.4. *Let Ω be $\mathcal{F}$-open set in $\mathbb{C}^n$ and u is bounded, the $\mathcal{F}$-plurisubharmonic function on Ω. Assume that for every $z \in \mathbb{C}^n$, there exists an Euclidean open neighbourhood $V_z \subset \mathbb{C}^n$ of z such that $u|_{V_z \cap \Omega}$ is $\mathcal{F}$-maximal on $V_z \cap \Omega$. Then, u is $\mathcal{F}$-maximal on Ω.*

Conclusion of Chapter 1

In this chapter, we present some of the necessary knowledge about the topological $\mathcal{F}$-topology, definitions and some properties of $\mathcal{F}$-$\mathcal{PSH}$ function and maximal $\mathcal{F}$-plurisubharmonic. At the same time also give, some of the results be used in the following chapters. In particular, the definition of the Monge-Ampère operator that is used to refer to the following two chapters, give Proposition 1.3.3, Proposition 1.3.4 (used in Chapter 2), and Proposition 1.1.9, Proposition 1.1.10, Proposition 1.2.5, Proposition 1.2.6 Proposition 1.2.7 (used in Chapter 3).

Chapter 2

Local property of maximal $\mathcal{F}$-plurisubharmonic function

In this chapter, we give sufficient conditions so that from $\mathcal{F}$-locally $\mathcal{F}$-maximal property of a $\mathcal{F}$-plurisubharmonic function on a $\mathcal{F}$-open set Ω in $\mathbb{C}^n$ also is $\mathcal{F}$-maximal on Ω. Concretely: Theorem 2.1.2 pointed out that with respect to a continuous $\mathcal{F}$-plurisubharmonic function on Ω, this function is $\mathcal{F}$-maximal on Ω if and only if it is $\mathcal{F}$-locally $\mathcal{F}$-maximal on Ω.

Next, we extend the above result by replacing the condition *continuous* in Theorem 2.1.2 by the weaker condition that is *bounded* of the $\mathcal{F}$-plurisubharmonic function on a $\mathcal{F}$-open set Ω in $\mathbb{C}^n$. Result achieved is Theorem 2.2.2 pointed out that $\mathcal{F}$-maximal property is equivalent with $\mathcal{F}$-locally $\mathcal{F}$-maximal property.

2.1 Local maximum of continuous $\mathcal{F}$-plurisubharmonic function

In this section, we will point out that *continuous* condition of a $\mathcal{F}$-plurisubharmonic function will ensure necessary and sufficient condition so that the $\mathcal{F}$-maximal $\mathcal{F}$-plurisubharmonic function is $\mathcal{F}$-locally $\mathcal{F}$-maximal.

Firstly, one has following definition:

Definition 2.1.1. Let $\Omega \subset \mathbb{C}^n$ is $\mathcal{F}$-open set and assume that $u \in \mathcal{F}\text{-}\mathcal{PSH}(\Omega)$. The function u is called *$\mathcal{F}$-locally $\mathcal{F}$-maximal* on Ω if for

every $z \in \mathbb{C}^n$, there exists a $\mathcal{F}$-open neighborhood $V_z \subset \mathbb{C}^n$ of z, such that $u|_{V_z \cap \Omega}$ is $\mathcal{F}$-maximal on $V_z \cap \Omega$.

The most important result of this section is the following theorem.

Theorem 2.1.2. *Let Ω be an $\mathcal{F}$-open set in $\mathbb{C}^n$. Assume that u is a continuous $\mathcal{F}$-plurisubharmonic function on Ω. Then u is $\mathcal{F}$-maximal on Ω if and only if it is $\mathcal{F}$-locally $\mathcal{F}$-maximal on Ω.*

2.2 Local maximum of bounded $\mathcal{F}$-plurisubharmonic function

In this section, we will point out "weaker" condition that is *bounded* of the $\mathcal{F}$-plurisubharmonic function to ensure necessary and sufficient condition so that the $\mathcal{F}$-maximal $\mathcal{F}$-plurisubharmonic function is $\mathcal{F}$-locally $\mathcal{F}$-maximal, that is extension in above result with new proving technique. This is one of results that carries a lot of scientific meaning, because it is true for $\mathcal{F}$-plurisubharmonic functions on $\mathcal{F}$-open sets. Moreover, we also point out a application with respect to a bounded $\mathcal{F}$-maximal $\mathcal{F}$-plurisubharmonic function.

First, we need to state and prove following important lemma.

Lemma 2.2.1. *Let D be a hyperconvex domain in $\mathbb{C}^n$ and let $\{\varphi_j\}$ be a sequence of plurisubharmonic functions defined on Euclidean open subsets U_j of D such that*

$$\bigcup_{j=1}^{\infty} U_j \Subset D.$$

Then, there exists an increasing sequence of bounded negative plurisubharmonic functions $\{\psi_k\}$ on D such that

(i) $\psi_k \to 0$ on $D \backslash E$, for some pluripolar set $E \subset D$;

(ii) φ_j is continuous on $U_j \cap \{\psi_k > -1\}$, for all $j, k \geq 1$.

The most important result of this section is the following theorem.

Theorem 2.2.2. *Let Ω be an $\mathcal{F}$-open set in $\mathbb{C}^n$ and let $u \in \mathcal{F}$-$\mathcal{PSH}(\Omega)$ be bounded. Then, the following statements are equivalent:*

(i) u is $\mathcal{F}$-maximal on Ω;

(ii) u is $\mathcal{F}$-locally $\mathcal{F}$-maximal on Ω;

(iii) $(dd^c u)^n = 0$ on $QB(\Omega)$.

Since above theorem, we give the example about classes of $\mathcal{F}$-maximal $\mathcal{F}$-plurisubharmonic functions through following corollary.

Corollary 2.2.3. *Let Ω be an $\mathcal{F}$-open set in $\mathbb{C}^n$ and let $u \in \mathcal{F}$-$\mathcal{PSH}(\Omega)$ be bounded. Then with $m \in \mathbb{N}^*$, we have*

$$u \circ \pi \in \mathcal{F}\text{-}\mathcal{MPSH}(\Omega \times \mathbb{C}^m),$$

where $\pi : \mathbb{C}^{n+m} \to \mathbb{C}^n$ is the canonical projection.

Conclusion of Chapter 2

In this chapter, we have presented the Theorem 2.1.2 and the Theorem 2.2.2. Specifically, a new proof technique has been proved, which demonstrates the equivalence of local and global maximal for $\mathcal{F}$-$\mathcal{PSH}$ functions on $\mathcal{F}$-open set in $\mathbb{C}^n$, with conditional *continuous* or *bounded* function . We have given the Lemma 2.2.1 and has used this Lemma to prove theorem 2.2.2. This chapter also gives an example for class of the $\mathcal{F}$-maximal $\mathcal{F}$-plurisubharmonic functions through the Corollary 2.2.3.

Chapter 3

Approximation of $\mathcal{F}$-plurisubharmonic function

In this chapter, we give sufficient conditions to a $\mathcal{F}$-plurisubharmonic function u on an Euclidean open subset Ω approximated by a increase sequence of plurisubharmonic functions. Where, approximation is understood in the sense of u can be approximated on $\overline{\Omega}$ by a increase sequence of smooth plurisubharmonic functions defined on Euclidean neighborhoods of Ω.

By giving concept of $\mathcal{F}$-hyperconvex domain and defining the class $\mathcal{F}$-$\mathcal{PSH}$ functions $\mathcal{E}_0(\Omega)$ and $\mathcal{F}_p(\Omega)$, we proved Theorem 3.3.1, where state that each function $u \in \mathcal{F}_p(\Omega)\,(p > 0)$ always approximate by an increase sequence of plurisubharmonic functions in neighborhood of Ω.

3.1 The class $\mathcal{E}_0$ of $\mathcal{F}$-plurisubharmonic functions

Firstly, we give the concept of $\mathcal{F}$-hyperconvex domain, definition the class $\mathcal{E}_0(\Omega)$ of $\mathcal{F}$-plurisubharmonic functions on Ω.

Definition 3.1.1. A bounded $\mathcal{F}$-domain Ω in $\mathbb{C}^n$ is called $\mathcal{F}$-*hyperconvex* if there exist a negative bounded plurisubharmonic function γ_Ω defined on a bounded hyperconvex domain Ω' such that

$$\Omega = \Omega' \cap \{\gamma_\Omega > -1\} \text{ and } -\gamma_\Omega \text{ is } \mathcal{F}\text{-plurisubharmonic on } \Omega.$$

We say that a bounded negative $\mathcal{F}$-plurisubharmonic function u defined on a bounded $\mathcal{F}$-hyperconvex domain Ω belongs to the $\mathcal{E}_0(\Omega)$ if $\int_\Omega (dd^c u)^n < +\infty$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\overline{\Omega \cap \{u < -\varepsilon\}} \subset \Omega' \cap \{\gamma_\Omega > -1 + \delta\}.$$

Proposition 3.1.2. *Let Ω be a bounded $\mathcal{F}$-hyperconvex domain in $\mathbb{C}^n$. Then $\mathcal{E}_0(\Omega) \neq \emptyset$.*

Note that, all bounded hyperconvex domain is $\mathcal{F}$-hyperconvex.
Now, we give a example to point out that there is bounded $\mathcal{F}$-hyperconvex domain Ω that has no point in Euclidean. Moreover, Ω satisfies the hypothesis of Theorem 3.3.1.

Exam 3.1.3.

Following, we have proposition.

Proposition 3.1.4. *Let Ω be a bounded $\mathcal{F}$-hyperconvex domain in $\mathbb{C}^n$. Assume that $u \in \mathcal{E}_0(\Omega)$ and $v \in \mathcal{F}\text{-}\mathcal{PSH}(\Omega)$ such that $u \leq v < 0$ on Ω. Then $v \in \mathcal{E}_0(\Omega)$ and*

$$\int_\Omega (-\rho)(dd^c v)^n \leq \int_\Omega (-\rho)(dd^c u)^n,$$

for every $\rho \in \mathcal{F}\text{-}\mathcal{PSH}^-(\Omega) \cap L^\infty(\Omega)$.

Moreover, if $u = v$ in $\{u > -\varepsilon_0\}$ for some $\varepsilon_0 > 0$ then,

$$\int_\Omega (dd^c v)^n = \int_\Omega (dd^c u)^n.$$

3.2 The class $\mathcal{F}_p$ of $\mathcal{F}$-plurisubharmonic functions

Firstly, we give the definition of the class $\mathcal{F}_p(\Omega)$ of $\mathcal{F}$-plurisubharmonic functions on Ω.

Definition 3.2.1. Let Ω be a bounded $\mathcal{F}$-hyperconvex domain in $\mathbb{C}^n$ and let $p > 0$. Denote by $\mathcal{F}_p(\Omega)$ the family of negative $\mathcal{F}$-plurisubharmonic

functions u defined on Ω such that there exist a decreasing sequence $\{u_j\} \subset \mathcal{E}_0(\Omega)$ that converges pointwise to u on Ω and

$$\sup_{j\geq 1} \int_\Omega (1+(-u_j)^p)(dd^c u_j)^n < +\infty.$$

Note that if $u \in \mathcal{F}_p(\Omega)$ then $u \in \mathcal{F}_q(\Omega)$ for all $q \in (0,p)$.

With those proper concepts given, we present some results through the following propositions.

Proposition 3.2.2. *Let Ω be a bounded $\mathcal{F}$-hyperconvex domain in $\mathbb{C}^n$ and let $p > 0$. Assume that $u \in \mathcal{F}_p(\Omega)$, $\{u_j\} \subset \mathcal{E}_0(\Omega)$ such that $u_j \searrow u$ on Ω and*

$$\sup_{j\geq 1} \int_\Omega (1+(-u_j)^p)(dd^c u_j)^n < +\infty.$$

Then

$$\int_{\{u>-\infty\}} (dd^c u)^n = \sup_{j\geq 1} \int_\Omega (dd^c u_j)^n.$$

Moreover, if u is bounded then

$$\int_\Omega (-v)(dd^c u)^n = \sup_{j\geq 1} \int_\Omega (-v)(dd^c u_j)^n,$$

for every $v \in \mathcal{F}\text{-}\mathcal{PSH}^-(\Omega) \cap L^\infty(\Omega)$.

Proposition 3.2.3. *Let Ω be a bounded $\mathcal{F}$-hyperconvex domain in $\mathbb{C}^n$ and let $p > 0$. Assume that $u \in \mathcal{F}_p(\Omega)$ and $v \in \mathcal{F}\text{-}\mathcal{PSH}(\Omega)$ with $u \leq v < 0$. Then $v \in \mathcal{F}_{\min(p,1)}(\Omega)$ and*

$$\int_{\{v>-\infty\}} (dd^c v)^n \leq \int_{\{u>-\infty\}} (dd^c u)^n.$$

Proposition 3.2.4. *Let Ω be a bounded $\mathcal{F}$-hyperconvex domain in $\mathbb{C}^n$ and let $p > 0$. Assume that $u \in \mathcal{F}_{\min(p,1)}(\Omega)$ and $v \in \mathcal{F}\text{-}\mathcal{PSH}^-(\Omega)$ such that*

$$(1+(-u)^p)(dd^c u)^n \leq (1+(-v)^p)(dd^c v)^n$$

on $\Omega \cap \{u > -\infty\} \cap \{v > -\infty\}$.
Then $u \geq v$ *on* Ω.

3.3 Approximation of $\mathcal{F}$-plurisubharmonic functions

In this section, we state and prove some proposition and main theorem of chapter that point out that every $\mathcal{F}$-plurisubharmonic function, that belong to the class $\mathcal{F}_p(\Omega)$, can be approximated by a increase sequence of negative plurisubharmonic functions defined on decrease sequence of hyperconvex domain that contains Ω.

Theorem 3.3.1. *Let* Ω *be a bounded* $\mathcal{F}$*-hyperconvex domain and let* $\{\Omega_j\}$ *be a decreasing sequence of bounded hyperconvex domains such that* $\Omega \subset \Omega_{j+1} \subset \Omega_j$*, for all* $j \geq 1$*. Assume that there exists* $\rho \in \mathcal{E}_0(\Omega)$*,* $\rho_j \in \mathcal{PSH}^-(\Omega_j)$ *with* $\rho_j \nearrow \rho$ *a.e. on* Ω*. Then, for every* $p > 0$ *and for every* $u \in \mathcal{F}_p(\Omega)$*, there exists an increasing sequence of functions* $u_j \in \mathcal{PSH}^-(\Omega_j)$ *such that* $u_j \to u$ *a.e. on* Ω*.*

Conclusion of Chapter 3

In this chapter, by giving the concept $\mathcal{F}$-hyperconvex domain, defines the class of $\mathcal{F}$-plurisubharmonic functions $\mathcal{E}_0(\Omega)$, $\mathcal{F}_p(\Omega)$, propositions, used complex Monge-Ampère operator and the propositions 1.1.9, clause 1.1.10, clause 1.2.5, clause 1.2.6, clause 1.2.7, have proved Theorem 3.3.1, which states that each of the function $u \in \mathcal{F}_p(\Omega)$ $(p > 0)$ is approximated by an increase sequence of the negative plurisubharmonic functions on decrease sequence of larger hypercovex domains

Conclusion and recommendation

I. Conclusion

Thesis has shown the conditions for the equivalence of property $\mathcal{F}$-locally $\mathcal{F}$-maximal and $\mathcal{F}$-maximal of the function $\mathcal{F}$-$\mathcal{PSH}$ and has shown sufficient conditions for a $\mathcal{F}$-plurisubharmonic function approximated by an increase sequence of the negative plurisubharmonic functions, concretely:

1. Thesis pointed out that with respect to a *continuous* $\mathcal{F}$-plurisubharmonic function on Ω, then this function is $\mathcal{F}$-maximal on Ω if and only if it is $\mathcal{F}$-locally $\mathcal{F}$-maximal on Ω (Theorem 2.1.2).

2. Thesis pointed out that with respect to a *bounded* $\mathcal{F}$-plurisubharmonic function on Ω, then this function is $\mathcal{F}$-maximal on Ω if and only if it is $\mathcal{F}$-locally $\mathcal{F}$-maximal on Ω (Theorem 2.2.2).

3. Thesis pointed out the conditions to establish approximation problem for the $\mathcal{F}$-plurisubharmonic function by increase sequence of the negative plurisubharmonicfunctions on decrease sequence of larger hypercovex domains (Theorem 3.3.1).

II. Recommendation

From resuts of thesis, we recommend some next research directions as follow:

1. Can we find out definition of complex Monge-Ampère operator for unbounded $\mathcal{F}$-$\mathcal{PSH}$ functions as Cegrell type?

2. Can we solve complex Monge-Ampre equation for $\mathcal{F}$-$\mathcal{PSH}$ function on $\mathcal{F}$-open domain? Can we study property of solution to that equation as: Stability of solution, continuity, Hölder continuity and so on?

3. Research some properties about weak $\mathcal{F}$-plurisubharmonic function and appply to pluripolar hulls.

Papers used in the thesis

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